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ON AUTO-OSCILLATIONS IN TWO-DIMENSIONAL NEAR-HAMILTONIAN DYNAMIC SYSTEMS

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A workable algorithm is proposed for reducing the Poincaré—Pontriagin generating equation which determines periodic solutions for small perturbations of two-dimensional Hamiltonian systems to the special (standard) form for the class of equations

$$x^{\cdot\cdot} + \alpha x + \beta x^3 = \varepsilon f(x, x^{\cdot}),\ \varepsilon \ll 1$$

where f is a polynomial. As an example of application, the problem of estimating the number of cycles, in particular of stable periodic solutions, i. e. of auto-oscillating modes, is considered. Results are illustrated on a specific example.

1. Let us consider the class of equations

$$x^{\cdot\cdot} + \alpha x + \beta x^3 = \varepsilon f(x, x^{\cdot})$$

or of equivalent form systems

$$x^{\cdot} = y,\ y^{\cdot} = -\alpha x - \beta x^3 + \varepsilon f(x, y) \tag{1.1}$$

$$f(x, y) = \sum_{j=1}^{m} \sum_{i=0}^{n} a_{ij} x^i y^j$$

where α and β are nonzero parameters, ε is a small parameter, and a_{ij} are constant coefficients.

An algorithm is proposed for reducing the Poincaré—Pontriagin generating equation to some special form. It was shown in [1] that a generating equation can be represented in the form of an integral of some expression dependent on perturbations and the periodic solution of the unperturbed input system. Integration is carried out along the unperturbed system closed trajectory free of singular points (equilibrium states).

Below, the input integral form of the generating equation is expressed in terms of elementary functions and complete elliptic integrals of the first and second kind. In spite of the different behavior of solutions of the unperturbed system, with various combinations of signs of α and β, the generating equation always reduces to the same form, which we shall call standard. In specific problems the standard form was directly calculated, for instance in [2].

We shall indicate all possible phase patterns of the unperturbed system $x^{\cdot} = y$, $y^{\cdot} = -\alpha x - \beta x^3$. In cases a) $\alpha > 0,\ \beta > 0$ and b) $\alpha > 0,\ \beta < 0$ we have a single cell filled by closed trajectories, and in cases c) $\alpha < 0,\ \beta > 0$

there are three cells in whose boundary pass two separatrix loops ("figure of eight"). Case d) $\alpha < 0$, $\beta < 0$ is of no interest.

In case a) $h \in (0, \infty)$ correspond to closed trajectories $y^2 / 2 + \alpha x^2 / 2 + \beta x^4 / 4 = h$, in case b) $h \in (0, -\alpha^2 / 4\beta)$ and in case c) $-h \in (-\alpha^2 / 4\beta, \infty)$. Values of $h \in (-\alpha^2 / 4\beta, 0)$ and $h \in (0, \infty)$ correspond to trajectories lying, respectively, in- and outside of the figure of eight.

Let $\Phi_{nm}^{*(1)}(h)$ be the expression for the standard form when $\beta > 0$, $h > 0$, $\Phi_{nm}^{*(2)}(h)$, when $\alpha > 0$, $\beta < 0$, $\Phi_{nm}^{*(3)}(h)$, and when $\alpha < 0$, $\beta > 0, -\alpha^2 / 4\beta \leqslant h < 0$.

T h e o r e m. The standard form of the generating equation of system (1.1) is

$$\Phi_{nm}^{*(s)}(k(h)) = \sum_j \Phi_{nj}^{(s)}(k(h)) \tag{1.2}$$

$$\Phi_{nj}^{(s)}(k(h)) = A_{nj}^{(s)}(k^2)\{P_{[n/2]+j}^{(s)}(k^2)\,\mathrm{K}(k) + Q_{[n/2]+j}^{(s)}(k^2)\,\mathrm{E}(k) + B_{nj}^{(s)}(k^2)\}, \quad s = 1, 2, 3 \tag{1.3}$$

where summation is carried out over odd j from unity to m; $A_{nj}^{(s)}$ and $B_{nj}^{(s)}$ are algebraic functions; $B_{nj}^{(1)} \equiv B_{nj}^{(2)} \equiv 0$, $B_{nj}^{(3)}$ depends only on coefficients $a_{2l+1,j}$, $l = 0, 1, 2, \ldots, [(n-1)/2]$, $n \geqslant 1$; $\mathrm{K}(k)$ and $\mathrm{E}(k)$ are complete elliptic integrals of the first and second kind, respectively; $k = k(h)$ is the modulus of elliptic integrals; $P_{[n/2]+j}^{(s)}(k^2)$ and $Q_{[n/2]+j}^{(s)}(k^2)$ are polynomials of power $[n/2] + j$ in k^2 which depends on coefficients $a_{2l,j}$, $l = 0, 1, 2, \ldots, [n/2]$, $n \geqslant 0$, and $[x]$ is the integral part of number x.

Proof of this theorem appears in Sect. 2, where an algorithm of constructing polynomials $P_{[n/2]+j}^{(s)}$ and $Q_{[n/2]+j}^{(s)}$, which does not require calculation of integrals, is also proposed.

2. Let us transform (1.1) to a more convenient form. In regions G filled by closed trajectories of the unperturbed system and separated from separatrices we pass from variables x, y to variables I and θ which represent action and angle, respectively, defined by

$$I = \frac{1}{2\pi}\oint y(x, h)\,dx, \quad \theta = \frac{\partial S(x, I)}{\partial I}$$

$$S(x, I) = \int_{x_0}^{x} y(x, h(I))\,dx$$

$$y(x, h) = [2(h - \alpha x^2/2 - \beta x^4/4)]^{1/2}$$

where integration is carried out along the closed trajectory of the unperturbed system, and x_0 is the coordinate of point x of that trajectory.

We represent the transformation $(x, y) \to (I, \theta)$ in the form

$$x = X(I, \theta), \quad y = Y(I, \theta) \tag{2.1}$$

where functions X and Y are periodic with respect to θ of period 2π. The substitution (2.1) transforms system (1.1) to

$$I^{\cdot} = \varepsilon\Phi(I, \theta), \quad \Phi(I, \theta) = f(X, Y)X_\theta' \tag{2.2}$$

$$\theta^{\cdot} = \omega(I) + \varepsilon Q(I, \theta), \quad \omega(I) = dh(I)/dI$$
$$Q(I, \theta) = -f(X, Y)X_I'$$

where Φ and Q are functions analytic in G and periodic with respect to θ of period 2π, and $\omega(I)$ is the frequency on closed trajectories of the unperturbed system. Note that in regions G $\omega(I) > 0$. Expanding function Φ in Fourier series

$$\Phi(I, \theta) = \sum_{k=-\infty}^{\infty} \Phi_k(I) \exp(ik\theta)$$

carrying out in (2.2) the substitution

$$I = u - \frac{i\varepsilon}{\omega} \sum_{k \neq 0} \frac{1}{k} \Phi_k(u) \exp(ik\theta)$$

and neglecting terms of order ε^2, we obtain the system

$$u^{\cdot} = \varepsilon\Phi_0(u), \quad \theta^{\cdot} = \omega(u) + 0(\varepsilon) \tag{2.3}$$

$$\Phi_0(u) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(u, \theta)\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(X, Y) X_\theta'\, d\theta$$

Thus the investigation of the input system with an accuracy to terms of order ε^2 reduces to the problem of investigation of the single differential equation

$$u^{\cdot} = \varepsilon\Phi_0(u) \tag{2.4}$$

The equation

$$\Phi_0(u) \equiv \Phi(u(h)) = 0 \tag{2.5}$$

which determines the equilibrium state of (2.4) is a generating equation. For fairly small $\varepsilon \neq 0$ the number of limit cycles in G does not exceed the number of real roots of Eq. (2.5).

It follows from (2.5) that the determination of $\Phi_0(h)$ requires the knowledge of the solution of the unperturbed system

$$x^{\cdot} = y \equiv \frac{\partial H}{\partial y}, \quad y^{\cdot} = -\alpha x - \beta x^3 \equiv -\frac{\partial H}{\partial x}, \quad H \equiv \frac{y^2}{2} + \frac{\alpha x^2}{2} + \frac{\beta x^4}{4}$$

From the integral $H = h$ we have

$$t - t_0 = \sqrt{2}\int_{x_0}^{x} \frac{dx}{[\beta(x_1^2 - x^2)(x^2 - x_2^2)]^{1/2}} \tag{2.6}$$

where x_1^2 and x_2^2 are the roots of equation

$$h - \alpha x^2/2 - \beta x^4/4 = 0$$

Setting $t_0 = 0$ and $x_0 = x_1$ with $\beta > 0$ and $x_0 = 0$ with $\alpha > 0, \beta < 0$, from (2.6) we obtain

$$\beta > 0,\ h > 0, \quad t = (2/\beta)^{1/2}(x_1^2 - x_2^2)^{-1/2} F(\varphi, k) \tag{2.7}$$
$$k = x_1(x_1^2 - x_2^2)^{-1/2}, \quad \cos\varphi = x/x_1$$
$$\alpha > 0,\ \beta < 0, \quad t = (-2/\beta)^{1/2} x_2^{-1} F(\varphi, k)$$
$$k = x_1/x_2, \quad \sin\varphi = x/x_1$$
$$\alpha < 0, \quad \beta > 0, \quad -\alpha^2/4\beta < h < 0, \quad t = (2/\beta)^{1/2} x_1^{-1} F(\varphi, k)$$

$$k = (x_1^2 - x_2^2)^{1/2} x_1^{-1}, \quad (1 - k^2 \sin^2\varphi)^{1/2} = x / x_1$$

where $F(\varphi, k)$ is an incomplete elliptic integral of the first kind. Using the known relations for Jacobi's functions, form (2.7) (with $\theta = \omega t$) we obtain

$$\begin{aligned}
&\beta > 0, \ h > 0, \ x(\theta) = x_1 \operatorname{cn}(2K\theta / \pi) \\
&\omega = \pi (\beta / 2)^{1/2} (x_1^2 - x_2^2)^{1/2} / (2K), \ x_1 = [2\alpha k^2 / (\beta (1 - 2k^2))]^{1/2} \\
&\alpha > 0, \ \beta < 0, \ x(\theta) = x_1 \operatorname{sn}(2K\theta / \pi) \\
&\omega = \pi (-\beta / 2)^{1/2} x_2 / (2K), \quad x_1 = [-2\alpha k^2 / (\beta (1 + k^2))]^{1/2} \\
&\alpha < 0, \ \beta > 0, \ h < 0, \ x(\theta) = x_1 \operatorname{dn}(K\theta / \pi) \\
&\omega = \pi (\beta / 2)^{1/2} x_1 / K, \quad x_1 = [-2\alpha / (\beta (2 - k^2))]^{1/2}
\end{aligned} \tag{2.8}$$

In accordance with (2.8) we distinguish three cases:

$$\begin{aligned}
&\Phi_0(u) = \Phi(k(h)) = \Phi_{nm}^{*(s)}(k), \quad s = 1, 2, 3. \\
&\beta > 0, \ h > 0, \ s = 1; \quad \alpha > 0, \ \beta < 0, \ s = 2 \\
&\alpha < 0, \ \beta > 0, \quad h < 0, \ s = 3
\end{aligned}$$

To prove the theorem we use the expression for Φ_0 in (2.3). First, we consider the case of even n and $j = 1$.

Let $\beta > 0$ and $h > 0$. In accordance with (2.3) we have

$$\Phi_{n1}^{(1)}(k) = \frac{x_1^2}{2\pi} \left(\frac{\alpha}{1 - 2k^2}\right)^{1/2} \sum_{i=0}^{n} a_{i1} x_1^{\,i} I_{i1} \tag{2.9}$$

$$I_{i1} = \int_0^{4K} [(1 - k^2) \operatorname{cn}^i \varphi + (2k^2 - 1) \operatorname{cn}^{i+2} \varphi - k^2 \operatorname{cn}^{i+4} \varphi] \, d\varphi$$

Properties of elliptic functions imply that $I_{i1} = 0$ when i is odd. Applying $l + 1$ times the recurrent formula [3]

$$\int_0^{4K} \operatorname{cn}^{m+4} \varphi \, d\varphi = \frac{(m+1)(1 - k^2)}{(m+3) k^2} \int_0^{4k} \operatorname{cn}^m \varphi \, d\varphi - \frac{(m+2)(1 - 2k^2)}{(m+3) k^2} \int_0^{4K} \operatorname{cn}^{m+2} \varphi \, d\varphi$$

we obtain the formula

$$I_{2l,1} = \frac{4}{k^{2l+2}} [(k^2 P_{2l+2}^{(1,1)}(k) - (1 - k^2) Q_{2l+2}^{(1,1)}(k)) K(k) + Q_{2l+2}^{(1,1)}(k) E(k)] \tag{2.10}$$

where $P_{2l+2}^{(1,1)}(k)$ and $Q_{2l+2}^{(1,1)}(k)$ are polynomials of power $2l + 2$ that contain only even powers of k. Hence we subsequently use the notation

$$\rho = k^2, \ P_{l+1}^{(1,1)}(\rho) \equiv P_{2l+2}^{(1,1)}(k), \ Q_{l+1}^{(1,1)}(\rho) \equiv Q_{2l+2}^{(1,1)}(k)$$

Since $K(k)$ and $E(k)$ can be represented in the form of power series for $0 \leqslant k < 1$ that contain only even powers of k, hence $K = K(\rho)$ and $E = E(\rho)$.

The calculation of $I_{2l,1}$ yields the following algorithm of derivation of polynomials $P_{l+1}^{(1,1)}(\rho)$ and $Q_{l+1}^{(1,1)}(\rho)$

$$P_{i+1}^{(1,1)} = \frac{2l-(2i-1)}{2l-(2i-3)}(1-\rho)\,Q_i^{(1,1)} \tag{2.11}$$

$$Q_{i+1}^{(1,1)} = \rho P_i^{(1,1)} - \frac{2l-(2i-2)}{2l-(2i-3)}(1-2\rho)\,Q_i^{(1,1)},\quad i = 1, 2, \ldots, l$$

$$P_1^{(1,1)}(\rho) = 2(1-\rho)/(2l+3),\quad Q_1^{(1,1)}(\rho) = (2\rho-1)/(2l+3)$$

From (2.10) we have

$$I_{2l,1} = (4/\rho^{l+1})\,[P_{l+2}^{(1,2)}(\rho)\,\mathrm{K} + Q_{l+1}^{(1,2)}(\rho)\,\mathrm{E}] \tag{2.12}$$

$$P_{l+2}^{(1,2)}(\rho) = \rho P_{l+1}^{(1,1)}(\rho) - (1-\rho)\,Q_{l+1}^{(1,1)}(\rho),\quad Q_{l+1}^{(1,2)}(\rho) = Q_{l+1}^{(1,1)}(\rho)$$

Polynomials $P_{l+2}^{(1,2)}$ and $Q_{l+1}^{(1,2)}$ have the following properties.

1°. $p_0 + q_0 = 0$, where p_0 and q_0 are the free terms of polynomials $P_{l+2}^{(1,2)}$ and $Q_{l+1}^{(1,2)}$, respectively, as implied by (2.12).

2°. $p_{l+2} = 0$, where p_{l+2} is the coefficient at the leading term of polynomial $P_{l+2}^{(1,2)}$.

Proof. From (2.12) we have $p_{l+2} = p_{l+1}^{(1)} + q_{l+1}^{(1)}$, where $p_{l+1}^{(1)}$ and $q_{l+1}^{(1)}$ are coefficients at the leading terms of polynomials $P_{l+1}^{(1,1)}$ and $Q_{l+1}^{(1,1)}$, respectively. If follows from (2.11) that $p_1^{(1)} + q_1^{(1)} = 0$ $(l = 0)$. Further proof is by induction. Let the condition $p_{l+2} = 0$ be satisfied for $l = i-1$, i.e. $p_i^{(1)} + q_i^{(1)} = 0$. We shall show that then $p_{i+1}^{(1)} + q_{i+1}^{(1)} = 0\ (i \leqslant n/2)$. Using (2.11) we obtain

$$p_{i+1}^{(1)} + q_{i+1}^{(1)} = -q_i^{(1)}\frac{2l-(2i-1)}{2l-(2i-3)} + p_i^{(1)} + 2\,\frac{2l-(2i-2)}{2l-(2i-3)}\,q_i^{(1)} = 0$$

Property 2° is proved.

On the strength of property 2 we substitute $P_{l+1}^{(1,2)}$ for $P_{l+2}^{(1,2)}$.

3°. $P_{l+1}^{(1,2)}(1) = 0$. This property follows from (2.11) and (2.12).

From (2.12) and (2.9) we have

$$\Phi_{n1}^{(1)}(\rho) = \frac{2}{\pi}\left(\frac{2}{\beta}\right)^{(n+2)/2}\left(\frac{\alpha}{1-2\rho}\right)^{(n+3)/2} F_{n1}^{(1)}(\rho) \tag{2.13}$$

$$F_{nj}^{(s)}(\rho) = P_{n/2+j}^{(s)}(\rho)\,\mathrm{K}(\rho) + Q_{n/2+j}^{(s)}(\rho)\,\mathrm{E}(\rho)$$

$$\{P_{n/2+j}^{(s)}, Q_{n/2+j}^{(s)}\} = \sum_{l=0}^{n/2} a_{2l,\,j} Z_s^{n/2-l}\{U_{l+j}^{(s)}, V_{l+j}^{(s)}\}$$

$$U_{l+j}^{(1)} = P_{l+j}^{(1,2)}(\rho),\quad V_{l+j}^{(1)} = Q_{l+j}^{(1,2)}(\rho)$$

$$U_{l+j}^{(2)} = P_{l+j}^{(2,2)}(\rho),\quad V_{l+j}^{(2)} = Q_{l+j}^{(2,2)}(\rho)$$

$$U_{l+j}^{(3)} = P_{l+j}^{(3,1)}(\rho),\quad V_{l+j}^{(3)} = Q_{l+j}^{(3,1)}(\rho)$$

$$Z_1 = \beta(1-2\rho)/(2\alpha),\ Z_2 = -\beta(1+\rho)/(2\alpha),\ Z_3 = -\beta(2-\rho)/(2\alpha)$$

In the second case, when $\alpha > 0$, $\beta < 0$, using (2.8) we similarly obtain

$$\Phi_{n1}^{(2)}(\rho) = \frac{2}{\pi}\left(-\frac{2}{\beta}\right)^{(n+2)/2}\left(\frac{\alpha}{1+\rho}\right)^{(n+3)/2} F_{n1}^{(2)}(\rho) \tag{2.14}$$

$$P_{l+1}^{(2,2)}(\rho) = \rho P_{l+1}^{(2,1)}(\rho) + Q_{l+1}^{(2,1)}(\rho),\quad Q_{l+1}^{(2,2)}(\rho) = -Q_{l+1}^{(2,1)}(\rho)$$

Polynomials $P_l^{(2,1)}(\rho)$, and $Q_{l+1}^{(2,1)}(\rho)$, $l = 0, 1, \ldots, n/2$ are determined with the use of the recurrent formulas

$$P_i^{(2,1)} = -\frac{2l-(2i-1)}{2l-(2i-3)}\,Q_i^{(2,1)}, \quad i = 1, 2, \ldots, l \tag{2.15}$$
$$Q_{i+1}^{(2,1)} = \rho P_{i-1}^{(2,1)}(\rho) + \frac{2l-(2i-2)}{2l-(2i-3)}(1+\rho)\,Q_i^{(2,1)}$$
$$P_0^{(2,1)}(\rho) = 2/(2l+3), \quad Q_1^{(2,1)}(\rho) = -(1+\rho)/(2l+3)$$

Since (2.15) implies that polynomials $P_{l+1}^{(2,2)}$ and $Q_{l+1}^{(2,2)}$ have the properties 1 and 2, hence in conformity with (2.13) the polynomials $P_{n/2+1}^{(2)}$ and $Q_{n/2+1}^{(2)}$ also have these properties.

In the third case, when $\alpha < 0,\ \beta > 0,\ -\alpha^2/4\beta \leqslant h < 0$, we have

$$\Phi_{n1}^{(3)}(\rho) = \frac{1}{2\pi}\left(\frac{2}{\beta}\right)^{(n+2)/2}\left(-\frac{\alpha}{2-\rho}\right)^{(n+3)/2} \times$$
$$\left[2F_{n1}^{(3)}(\rho) \pm \pi\sqrt{Z_3(\rho)}\,R_{n/2+1}(\rho)\right]$$
$$R_{n/2+1}(\rho) = \sum_{l=0}^{n/2-1} a_{2l+1,1} Z_3^{n/2-(l+1)}(\rho)\left(P_{l+1}(\rho) + \frac{2-\rho}{2}\,Q_{l+1}(\rho)\right)$$

where the plus sign corresponds to the region lying inside the right-hand separatrix loop $(x > 0)$ and the minus sign to the region inside the left-hand separatrix loop $(x < 0)$. The polynomials $P_{l+1}^{(3,1)}(\rho)$ and $Q_{l+1}^{(3,1)}(\rho)$ are determined using the recurrent formulas

$$P_{i+1}^{(3,1)} = \frac{2l-(2i-1)}{2l-(2i-3)}(\rho-1)\,Q_i^{(3,1)}, \quad i = 1, 2, \ldots, l$$
$$Q_{i+1}^{(3,1)} = P_i^{(3,1)} + \frac{2l-(2i-2)}{2l-(2i-3)}(2-\rho)\,Q_i^{(3,1)}$$
$$P_1^{(3,1)}(\rho) = 2(\rho-1)/(2l+3), \quad Q_1^{(3,1)}(\rho) = (2-\rho)/(2l+3)$$

and posess properties 1 and 3.

The polynomials $P_{l+1}(\rho)$ and $Q_{l+1}(\rho)$ in the expression for $R_{n/2+1}(\rho)$ are determined by the recurrent formulas

$$P_{i+1} = \frac{l-i+1}{l-i+2}(\rho-1)\,Q_i$$
$$Q_{i+1} = P_i + \frac{2(l-i)+3}{2(l-i)+4}(2-\rho)\,Q_i, \quad i = 1, 2, \ldots, l$$
$$P_1(\rho) = (\rho-1)/(l+2), \quad Q_1(\rho) = (2-\rho)/2(l+2)$$

Derivation of the standard form for $j > 1$ does not in principle differ from that for $j = 1$. The final results are as follows.

In the case of $\beta > 0,\ h > 0$ we have

$$\Phi_{nj}^{(1)}(\rho) = \frac{2}{\pi}\left(\frac{2}{\beta}\right)^{(n+j+1)/2}\left(\frac{\alpha}{1-2\rho}\right)^{(n+2j+1)/2} F_{nj}^{(1)}(\rho) \tag{2.16}$$

$$\{P_{l+j}^{(1,2)}(\rho),\ Q_{l+j}^{(1,2)}(\rho)\} = \sum_{r,\,k=0}^{(j+1)/2} f_{rk}\{P_b^{(1,1)}(\rho),\ Q_{b-1}^{(1,1)}(\rho)\} + \tag{2.17}$$
$$\{L_{lj} + M_{lj},\ N_{lj} + S_{lj}\}, \quad b = l + r + k,\ b \geqslant 2$$

$$f_{rk} = (-1)^r C^r_{(j+1)/2} C^k_{(j+1)/2} \rho^{(j+1)/2-r} (1-\rho)^{(j+1)/2-k}$$
$$L_{lj} = \delta_{l0} [(1 + j - 2j\rho)\, \rho^{(j-1)/2} (1-\rho)^{(j+1)/2} / 2]$$
$$N_{lj} = \delta_{l0} [(1 + j)(2\rho - 1)\, \rho^{(j-1)/2} (1-\rho)^{(j-1)/2} / 2]$$
$$M_{lj} = \delta_{l1} [-\rho^{(j+1)/2} (1-\rho)^{(j+3)/2}], \quad S_{lj} = \delta_{l1} [\rho^{(j+1)/2} (1-\rho)^{(j+1)/2}]$$
$$P_b^{(1,1)}(\rho) = \rho P_{b-1}^{(1,3)}(\rho) - (1-\rho)\, Q_{b-1}^{(1,3)}(\rho), \quad Q_{b-1}^{(1,1)}(\rho) \equiv Q_{b-1}^{(1,3)}(\rho)$$

where δ_{li} is the Kronecker delta. Expressions for $P_{b-1}^{(1,3)}(\rho)$ and $Q_{b-1}^{(1,3)}(\rho)$ are determined by the recurrent formulas

$$P_{i+1}^{(1,3)} = \frac{2b-(2i+3)}{2b-(2i+1)} (1-\rho)\, Q_i^{(1,3)}, \quad b > 2$$
$$Q_{i+1}^{(1,3)} = \rho P_i^{(1,3)} - \frac{2b-(2i+2)}{2b-(2i+1)} (1-2\rho)\, Q_i^{(1,3)}, \quad i = 1, 2, \ldots, b-2$$
$$P_1^{(1,3)}(\rho) = \frac{2b-3}{2b-1}(1-\rho), \quad Q_1^{(1,3)}(\rho) = \frac{2b-2}{2b-1}(2\rho-1)$$

This yields the method for transforming the formulas obtained above for $j = 1$ to suit the case of $j > 1$. Thus for $\alpha > 0$, $\beta < 0$ we have

$$\Phi_{nj}^{(2)}(\rho) = \frac{2}{\pi}\left(-\frac{2}{\beta}\right)^{(n+j+1)/2} \left(\frac{\alpha}{1+\rho}\right)^{(n+2j+1)/2} F_{nj}^{(2)}(\rho) \tag{2.18}$$

and for $\alpha < 0$, $\beta > 0$, $h < 0$

$$\Phi_{nj}^{(3)}(\rho) = \frac{1}{2\pi}\left(\frac{2}{\beta}\right)^{(n+j+1)/2} \left(-\frac{\alpha}{2-\rho}\right)^{(n+2j+1)/2} \times [2F_{nj}^{(3)} \pm \pi \sqrt{Z_3(\rho)}\, R_{n/2+j}(\rho)] \tag{2.19}$$

Using the notation $B_{nj}^{(3)}(\rho) \equiv \pm \pi \sqrt{Z_3(\rho)}\, R_{n/2+j}(\rho)$, from (2.16), (2.18), and (2.19) we obtain formula (1.3). For even j we have $\Phi_{nj}^{(s)}(\rho) \equiv 0$, $s = 1, 2, 3$. Polynomials $P_{n/2+j}^{(s)}$, and $Q_{n/2+j}^{(s)}$, $s = 1, 2, 3$ also posess properties 1 and 3.

The case of odd n is similary considered. To do this it is sufficient to substitute, retaining the notation $\Phi_{nj}^{(s)}$, $A_{nj}^{(s)}$, and $B_{nj}^{(s)}$, $s = 1, 2, 3$, in respective expressions $n-1$ for n and carry out summation in $R_{(n-1)/2+j}$ with respect to l from 0 to $(n-1)/2$. The theorem is proved.

3. In accordance with (2.16) and (2.18) the problem of obtaining an estimate of the number of real roots of the generating equation $\Phi_{nj}^{(s)}(\rho) = 0$, $s = 1, 2$ reduces to that of estimating the number N of real roots of the equation

$$F_{nj}^{(s)}(\rho) = 0, \quad 0 \leqslant \rho < 1, \quad s = 1, 2 \tag{3.1}$$

In the case of $s = 3$ the problem reduces to estimating the number of real roots of the equation

$$2F_{nj}^{(3)}(\rho) \pm \pi \sqrt{Z_3(\rho)}\, R_{n/2+j}(\rho) = 0 \tag{3.2}$$

When $a_{2l+1,j} = 0$, $l = 0, 1, 2, \ldots, n/2 - 1$, Eq. (3.2) assumes the form $F_{nj}^{(3)}(\rho) = 0$, and when $a_{2l,j} = 0$, $l = 0, 1, 2, \ldots, n/2$ it reduces to the algebraic equation $R_{n/2+j}(\rho) = 0$. In the latter case $N \leqslant n/2 + j$.

Let us estimate the number N_1 of double roots of equation $F_{nj}^{(s)}(\rho) = 0$, $0 <$

$\rho < 1$, $s = 1, 2, 3$. Note that function $F_{nj}^{(s)}$ is analytic when $0 \leqslant \rho < 1$.
Differentiation of $F_{nj}^{(s)}(\rho)$ yields

$$d^m F_{nj}^{(s)}(\rho) / d\rho^m = P_c^{(s)} \mathrm{K} + Q_c^{(s)} \mathrm{E}, \quad m = 1, 2, \ldots \qquad (3.3)$$

$$P_c^{(s)} \equiv \frac{dP_{c+1}^{(s)}}{d\rho} - \frac{1}{2\rho}(P_{c+1}^{(s)} + Q_{c+1}^{(s)})$$

$$Q_c^{(s)} \equiv \frac{dQ_{c+1}^{(s)}}{d\rho} + \frac{1}{2\rho}\left(\frac{1}{1-\rho} P_{c+1}^{(s)} + Q_{c+1}^{(s)}\right)$$

$$c = n/2 + j - m$$

The double roots of equation $F_{nj}^{(s)}(\rho) = 0$, $s = 1, 2, 3$ satisfy the system

$$F_{nj}^{(s)}(\rho) = 0, \; F_{n,j-1}^{(s)}(\rho) = 0 \qquad (3.4)$$

Multiplying the first equation of this system by $P_{n/2+j-1}^{(s)}$, the second by $P_{n/2+j}^{(s)}$ and subtracting the first from the second, we obtain the system

$$F_{nj}^{(s)}(\rho) = 0, \quad P_{n/2+j}^{(s)}(\rho)\, Q_{n/2+j-1}^{(s)}(\rho) - P_{n/2+j-1}^{(s)}(\rho)\, Q_{n/2+j}^{(s)}(\rho) = 0 \qquad (3.5)$$

The number of nonzero roots of system (3.4) does not exceed the number of nonzero roots of system (3.5). Since on the strength of properties 1 and 3 and formulas (3.3) $P_{n/2+j-1}^{(s)}$ and $Q_{n/2+j-1}^{(s)}$ are polynomials of power $n/2 + j - 1$, hence the second of equations of system (3.5) is a polynomial of power $n + 2j - 1$.

It follows from (2.13) and (2.17) that

$$2p + q = 0, \quad p_0^{(s)} q_{01}^{(s)} - p_{01}^{(s)} q_0^{(s)} = 0 \qquad (3.6)$$

where p and q are coefficients at leading terms of polynomials $P_{n/2+j}^{(1)}$ and $Q_{n/2+j}^{(1)}$ respectively; $p_0^{(s)}$ and $q_0^{(s)}$ are the free terms of polynomials $P_{n/2+j}^{(s)}$ and $Q_{n/2+j}^{(s)}$; $p_{01}^{(s)}$ and $q_{01}^{(s)}$ are the free terms of polynomials $P_{n/2+j-1}^{(s)}$ and $Q_{n/2+j-1}^{(s)}$, respectively, and $s = 2, 3$. When $s = 1$, the coefficient at the leading term of the second of Eqs. (3.5) is by virtue of (3.6) zero, and when $s = 2, 3$, the free term vanishes. Thus the estimate $N_1 \leqslant n + 2(j - 1)$ is correct. From this we obtain the estimate $N \leqslant 2N_1 + 1$ when $s = 1, 2$. In the quasilinear case with $\beta = 0$ we have $N \leqslant n/2 + j - 1$.

Note that $d\Phi_{nj}^{(s)}(\rho_*) / d\rho = F_{n,j-1}^{(s)}(\rho_*)$, where ρ_* is the root of the generating equation. If $\varepsilon F_{n,j-1}^{(s)}(\rho_*)\,(d\rho / dI) > 0 \; (< 0)$, the state of equilibrium $u = u(\rho_*)$ of Eq. (2.4) is unstable (stable).

As an example we consider the equation

$$x^{\cdot\cdot} + \alpha x + \beta x^3 = \varepsilon (a_{01} + a_{11} x + a_{21} x^2) x^{\cdot} \qquad (3.7)$$

Using the expression for $\Phi_{21}^{(s)}$, $s = 1, 3$ and the investigations of behavior of solutions in the neighborhood of separatrices of the unperturbed system, we establish the over-all topological structure of behavior of the solution of Eq. (3.7) when $\alpha < 0$, $\beta > 0$. We assume that $a_{11} \neq 0$, $a_{01} = 1$, $\alpha = -1$, and $\beta = 1$. The case of $a_{11} = 0$ with $\alpha > 0$ was considered in [4], and for any arbitrary $\alpha \neq 0$ in [5, 6]. Let us represent (3.7) in the form

$$x^{\cdot} = P(x, y) = y, \quad y^{\cdot} = Q(x, y, \varepsilon) \qquad (3.8)$$

$$Q = x - x^3 + \varepsilon (1 + a_{11} x + a_{21} x^2) y$$

and investigate the behavior of solutions in the small neighborhood of the unperturbed separatrix. Note that since the saddle parameter $\sigma_c \equiv P_x' + Q_y' |_{x=y=0} = \varepsilon \neq 0$, hence not more than one limit cycle can emerge from the separatrix loop for any

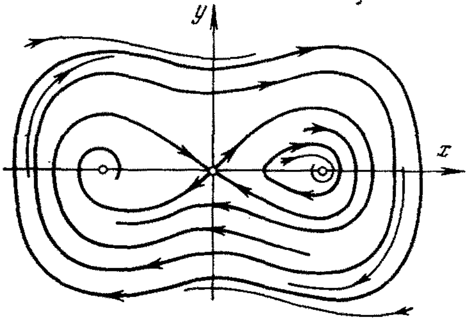

Fig. 1

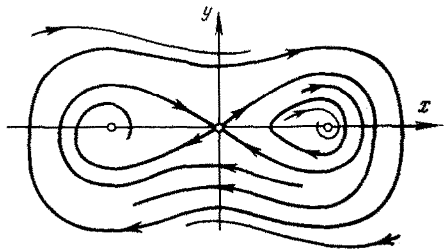

Fig. 2

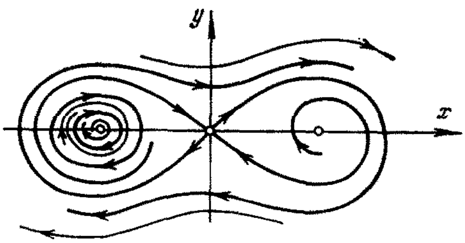

Fig. 3

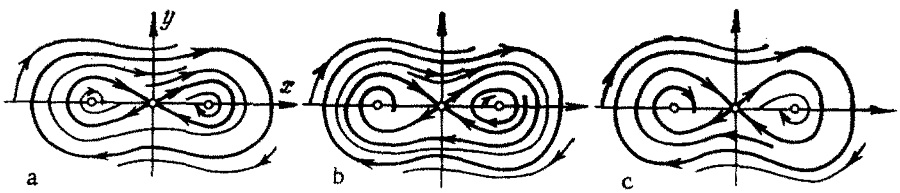

Fig. 4

fixed $\varepsilon \neq 0$ [7]. To solve the problem of relative position of separatrices we use the results of investigations [8]. We denote by $\varepsilon\Delta_1^+$ the parameter which defines with an accuracy to terms of order ε^2 the distance between the respective stable and unstable separatrices in the region of $x > 0$, and by $\varepsilon\Delta_1^-$ in the region of $x < 0$. According to [8] we have

$$\Delta_1^{\pm} = \int_{-\infty}^{\infty} [1 + a_{11}x_0(t) + a_{21}x_0^2(t)]\, y_0^2(t)\, dt$$

where $x_0(t)$, $y_0(t)$ is the solution of the unperturbed system on the separatrix.

From (2.8) we have $x_0(t) = \pm\sqrt{2}\,(1/\operatorname{ch} t)$, $y_0(t) = \mp(\operatorname{sh} t/\operatorname{ch}^2 t)$, $k = 1$, hence

$$\Delta_1^{\pm} = 2\left[\frac{2}{3} \pm \frac{\pi}{8}\sqrt{2}\,a_{11} + \frac{8}{15}\,a_{21}\right]$$

Using equations $\Delta_1^{\pm} = 0$ we determine

$$a_{11}^{+} = -a_{11}^{-} = -16\,(5 + 4a_{21})\,/\,(15\pi\sqrt{2})$$

When $a_{11} = a_{11}^{+} + O(\varepsilon)$, the solutions of system (3.8) have the right-hand loop of the separatrix, and when $a_{11} = a_{11}^{-} + O(\varepsilon)$ its left-hand loop.

Investigation of solutions in regions inside the unperturbed separatrix (figure of eight) is carried out using the standard form $\Phi_{21}^{(3)}$ and outside it using $\Phi_{21}^{(1)}$.

The real roots of equations $\Phi_{21}^{(s)}(\rho) = 0$, $s = 1,3$ that determine h and the corresponding closed curves of the unperturbed system from which emerge limit cycles for small $\varepsilon \neq 0$, depend on parameters a_{11} and a_{21}. It is, therefore, possible to divide the plane a_{11}, a_{21} in two regions that correspond to different numbers of limit cycles in the perturbed system. It is established that for Eq. (3.7) the maximum number of limit cycles is three.

The bifurcation boundaries, which are generally determined with an accuracy to terms of order ε, divide the plane a_{11}, a_{21} in 32 regions. The most typical rough phase patterns of system (3.8) are plotted in Figs. 1—3. Part of the finer topological structures appear in Fig. 4.

Although in the symmetric case, when $a_{11} = 0$, we have $\Phi_{21}(1) = \Phi_{21}^{(3)}(1)$ and the generating equation actually determines limit cycles up to the separatrix (figure of eight in Fig. 4, a), it is not so in the asymmetric case, when $a_{11} \neq 0$, because of $\Phi_{21}^{(1)}(1) \neq \Phi_{21}^{(3)}(1)$. When $a_{11} \neq 0$ a limit cycle emerges from the separatrix loop of the perturbed system (Fig. 4, b). This property cannot be established by the method of small parameter.

Fig. 4, c corresponds to the fine structure that has a separatrix loop and a double limit cycle. Transition from Fig. 1 to Fig. 2 takes place through the fine structure shown in Fig. 4, b (4, a).

REFERENCES

1. Pontriagin, L. S., On near-Hamiltonian dynamic systems. ZhETF. Vol. 4, No. 9, 1934.

2. Bautin, N. N., On approximations and the coarseness of the parameter space of a dynamic system. PMM, Vol. 33, No. 6, 1969.

3. Gradshtein, I. S. and Ryzhik, I. M., Tables of Integrals, Sums, Series, and Products. Moscow, "Nauka", 1971.

4. Morozov, A. D. and Shil'nikov, L. P., The mathematical theory of oscillation synchronization. Dokl. Akad. Nauk SSSR, Vol. 223, No. 6, 1975.

5. Arnol'd, V. I., Loss of oscillation stability near resonance and versal deformations of equivariant vector fields. Funktsional'nyi Analiz i ego Prilozheniia, Vol. 11, No. 2, 1977.

6. Il'iashenko, Iu. S., On the zeros of special Abel integrals in the real region. Funktsional'nyi Analiz i ego Prilozheniia, Vol. 11, No. 4, 1977.

7. Andronov, A. A., Leontovich, E. A., Gordon, I. I., and Maier, A. G., The theory of Bifurcation of Dynamic Systems in a Plane. Moscow, "Nauka", 1967.

8. Mel'nikov, V. K., On stability of the center subjected to time periodic perturbations. Tr. Mosk. Matem. Obshch., Vol. 12, 1963.

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